Numerical solution of radiative transfer equation coupled with nonlinear heat conduction equation

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Abstract *A new way of solving the steady-state coupled radiative-conductive problem in semi-transparent media is proposed. An angular discretization technique is applied in order to express the radiative transfer equation (RTE) in an inhomogeneous system of linear differential equations associated with Dirichlet boundary conditions. The system is solved by a direct method, after diagonalizing the characteristic matrix of the medium. The RTE is coupled with the nonlinear heat conduction equation. A simulation of a real semi-transparent medium composed of silica fibers is illustrated. Comparison with results of other methods validates the new model. Moreover, the general scheme is easy to code and fast. The algorithm proved to be robust and stable.*

Nomenclature

y = distance, m
E = medium thickness, m
T = temperature, K
T_o = boundary temperature at the abscissa 0, K
T_E = boundary temperature at the abscissa E, K
L_λ = spectral radiation intensity, W/(m^3·sr)
L_λ^0 = spectral radiation intensity emitted by a black body, W/m^3
P_λ^* = spectral phase function
Q_r = heat flux by radiation, W/m^2
Q_c = heat flux by conduction, W/m^2
Q_t = total heat flux, W/m^2
S_r = radiative term source
λ = wavelength, μm
λ_c = thermal conductivity, W/m·K
$\sigma_{a\lambda}$ = Spectral absorption coefficient, m^{-1}
$\sigma_{s\lambda}$ = spectral scattering coefficient, m^{-1}
$\sigma_{e\lambda}$ = spectral extinction coefficient, m^{-1}
$\sigma_s P_\lambda^*$ = scattering geometry
μ = angular direction
ρ = medium density, kg/m^3

Subscripts
a = absorption
s = scattering
e = extinction
r = radiative
c = conductive
t = total
λ = monochromatic

Superscripts
+, − = in the positive or negative y-direction
o = relating to the black body

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1. Introduction

Radiative heat transfer coupled with conduction through semi-transparent media is the subject of a large number of studies because of its multiple practical applications, especially in thermal insulation. Many theoretical studies have been devoted to the solution of RTE in semi-transparent absorbing and isotropic scattering media (Kelley, 1996; Banoczi and Kelley, 1998; Larsen and Nelson, 1982; Pitkaranta and Scott, 1983) and recently, for a non-grey participating medium with anisotropic scattering (Boulet *et al*., 1993). The works carried out in our laboratory (Boulet *et al*., 1993; Guilbert *et al*., 1987) give access to the three coefficients characterising the radiative transfer: the spectral absorption coefficient $\sigma_{a\lambda}(\mu)$, the spectral diffusion coefficient $\sigma_{s\lambda}(\mu)$ and the spectral phase function $P^*_\lambda(\mu' \rightarrow \mu)$. These coefficients intervene in the RTE. In particular, a computer program has been developed for the calculation of these coefficients.

Boulet *et al*. (1993) studied, within this context, the combined radiative and conductive heat transfer in a medium with axial symmetry and composed of silica fibers randomly oriented in planes parallel to the boundaries. These latter are held at fixed temperatures. These authors used a model known as "two-streams" in which the RTE is formulated in an inhomogeneous system of linear differential equations associated with Dirichlet boundary conditions. A convenient method for solving homogeneous systems is a matrix exponential method, but a direct application is numerically unstable. Therefore, Boulet *et al*. (1993) solved the homogeneous system using two characteristic matrices representative of transmission and reflection by a layer of given thickness (Waterman, 1981; Flateau and Stephens, 1988). These matrices were computed by a process called the "doubling" and "adding" method. After initialization of the starting matrix for a layer of very small thickness, it consists of calculating transmission and reflection matrices for layers of increasingly significant thickness in an iterative way. A particular solution of the system is obtained in polynomial form. This present paper continues this research work. The medium is non-grey, hence the RTE is a spectral equation, which must be solved over the whole spectrum. A fast method thus needs to be developed in order to solve this equation because of the long computing times. The previous method gives satisfactory results but the computing times are very long, primarily related to the initialization of the transmission and reflection matrices. Moreover, it cannot be extended to the transient-state because of the particular solution. In the transient-state, there can be very fast variations of temperature and in this case, one cannot seek a suitable particular solution in polynomial form. A preceding study led us to solve the system by a backward-forward finite difference scheme (Asllanaj *et al*., 2000). This method was found to be satisfactory based on the results and speed in the case of the steady-state using a uniform mesh.

In section 2, we present the equations governing the simultaneous steady-state radiative-conductive heat transfer, in a semi-transparent medium. In section 3, we present the angular discretization technique that allows expressing the RTE as an inhomogeneous system of linear differential equations. We show then how the diagonalizing procedure of the characteristic matrix of the medium makes it possible to circumvent the numerical instability problem. In section 4, the method is tested on a real semi-transparent medium and results are compared with those obtained by preceding models. Finally, in the last section, we discuss the special interest of this method in the transient-state.

2. Formulation of the problem

We consider the equations for radiative-conductive heat transfer through a planar fibrous medium. The domain is supposed to be homogeneous. The unknowns are the radiation intensity $L_\lambda(y,\mu)$ referred to the wavelength λ, at a point y, in the direction μ and the temperature $T(y)$ at position y. For a non-grey, absorbing, emitting and anisotropically scattering medium of thickness E, assuming transfer in the y-direction, with an axial symmetry, the RTE, as described in Modest (1993), Siegel and Howell (1992) and Ozisik (1973), is

$$\mu \cdot \frac{\partial L_\lambda(y,\mu)}{\partial y} = \sigma_{a\lambda}(\mu)\cdot L_\lambda^o(T(y)) - \sigma_{e\lambda}(\mu)\cdot L_\lambda(y,\mu) + \frac{1}{2}\cdot\int_{-1}^{1}\sigma_s P_\lambda^*(\mu' \to \mu)\cdot L_\lambda(y,\mu')d\mu' \tag{1}$$

for $0 < y < E$, $\mu \epsilon [-1,1]\{0\}, \lambda > 0$. μ is the cosine of the polar angle between the directions of propagation and transfer. In this equation, the terms on the right hand side describe respectively internal emission, extinction phenomena and the intensity of the scattering in the μ-direction. $L_\lambda^o(T(y))$ is the monochromatic intensity of the black body at temperature T, given by Planck's law as

$$L_\lambda^0(T) = \frac{C_1}{\lambda^5\cdot\left[\exp(\frac{C_2}{\lambda\cdot T}) - 1\right]} \tag{2}$$

where C_1 and C_2 are two constants of the radiation

$$C_1 = 1.19\cdot 10^{-16}\mathrm{W/m^2} \ \text{ and } C_2 = 1.4388\cdot 10^{-2}\mathrm{m\cdot K} \tag{3}$$

The monochromatic extinction coefficient is $\sigma_{e\lambda} = \sigma_{a\lambda} + \sigma_{s\lambda}$

The coefficient $\sigma_s P_\lambda^*$ represents the scattering geometry and is defined by

$$\sigma_s P_\lambda^*(\mu' \to \mu) = \sigma_{s\lambda}(\mu')\rho_\lambda(\mu' \to \mu)\ \forall \mu, \mu' \epsilon [-1,1]\backslash\{0\} \tag{4}$$

where P_λ^* is the spectral phase function. The function $\mu \to \frac{1}{2}\cdot P_\lambda^*(\mu' \to \mu)$ is a probability density on [–1,1]\{0} which verifies

$$\frac{1}{2}\cdot\int_{-1}^{1} P_\lambda^*(\mu' \to \mu)\, d\mu = 1 \tag{5}$$

In addition, coefficients $\sigma_{a\lambda}$ $\sigma_{e\lambda}$ and $\sigma_s P_\lambda^*$ in (1) are strictly positive.

The boundary surfaces at $y = 0$ and $y = E$ are both black surfaces kept at uniform temperatures T_o and T_E respectively. Then the radiative boundary conditions (see (Ozisik, 1973)) are

$$\begin{aligned} L_\lambda(0,\mu) &= L^0_\lambda(T_o) \quad \text{for} \quad 0 < \mu \le 1 \\ L_\lambda(E,\mu) &= L^0_\lambda(T_E) \quad \text{for} \quad -1 \le \mu < 0 \end{aligned} \tag{6}$$

The temperature T satisfies the nonlinear energy equation

$$-\frac{d}{dy}(\lambda_c(T(y)) \cdot \frac{dT}{dy}(y)) = S_r(y) \tag{7}$$

for $0 < y < E$ with Dirichlet boundary conditions

$$T(0) = T_o, \; T(E) = T_E \tag{8}$$

and coupling to the radiative transfer by the radiative source term

$$S_r(y) = -\frac{d}{dy}(Q_r(y)) \tag{9}$$

with

$$Q_r(y) = 2\pi \cdot \int_{\lambda=0}^{\infty} \int_{\mu=-1}^{1} L_\lambda(y,\mu) \cdot \mu \;\; d\mu \cdot d\lambda \tag{10}$$

Q_r is the radiative heat flux and $L_\lambda(y,\mu)$ is determined from the solution of the radiative transfer equation (1). The conductive heat flux is defined by

$$Q_c(y) = -\lambda_c(T(y)) \cdot \frac{dT}{dy}(y) \quad 0 < y < E \tag{11}$$

In equations (7) and (11), $\lambda_c(T(y))$ is the thermal conductivity of the medium which depends on temperature. The heat transfer by conduction was calculated from Fourrier's law using a semi-empirical expression for the conductivity, developed for fibrous insulators made of silica fibers. It is based on experimental data obtained from a guarded hot plates apparatus at the Saint Gobain Research Center (Langlais and Klarsfeld, 1985)

$$\lambda_c(T) = a \cdot T^{0,81} + b \cdot T + c \quad (\text{mW/m} \cdot \text{K}) \tag{12}$$

where $a = 0.2572$; $\mathrm{c} = 0.0527 \cdot \rho^{0.91}$ $\mathrm{b} = 0.0013 \cdot \mathrm{c}$ and T is the medium temperature (K) and ρ the medium density (kg/m^3).

In addition, thermal conductivity being strictly positive whatever the temperature, equation (7) is not degenerate.

Total heat flux is given by the sum of radiative and conductive flux

$$Q_t = Q_r + Q_c \tag{13}$$

From equations (7), (9), (11) and (13), we have

$$\frac{d}{dy}(Q_t(y)) = 0 \quad \forall\, 0 < y < E \tag{14}$$

We point out that the numerical method used to solve the coupled problem, that we detail in the following, will lead to a solution with this property.

3. Resolution of the radiative transfer equation

In this section, we are interested in the RTE discretization which involves three distinct problems:

(1) angular discretization;

(2) space discretization; and

(3) spectral discretization.

3.1 Angular discretization

Space is divided into m sectors and $\{\mu_i\}_{i=1}^{m}$ denote the discrete angular directions where $0 < \mu_i \leq 1$ for $1 \leq i \leq m/2$ and $\mu_i = -\mu_{m+1-i}$ for $m/2 + 1 \leq i \leq m$. The resulting angular discretization of the RTE led us to write the following linear differential system as (Asllanaj *et al.*, 2000):

$$\frac{dL_\lambda(y)}{dy} = A_\lambda \cdot L_\lambda(y) + \mathscr{E}_y^o(y) \tag{15}$$

where L_λ is the spectral radiation intensity vector of dimension m formed of the two vectors L_λ^+ and L_λ^-

$$L_\lambda(y) = \begin{bmatrix} L_\lambda^+(y) \\ L_\lambda^-(y) \end{bmatrix} \tag{16}$$

L_y^+ and L_λ^- are two radiation intensity fields of dimension m/2. They are linked to the "front" $0 < \mu \leq 1)$ and "back" $(-1 \leq \mu < 0)$ hemispheres (Figure 1)

$$L_\lambda^+(y) = [L_\lambda(y,\mu_i)]_{1\leq i\leq m/2} \text{ and } L_\lambda^-(y) = [L_\lambda(y,-\mu_i)]_{1\leq i\leq m/2} \tag{17}$$

$\mathscr{E}_\lambda^0(y)$ is the emission vector specific to the medium of dimension m defined by

$$\mathscr{E}_\lambda^o(y) = \begin{bmatrix} \mathscr{E}_\lambda^{o,+}(y) \\ -\mathscr{E}_\lambda^{o,+}(y) \end{bmatrix} \tag{18}$$

where $\mathscr{E}_\lambda^{o,+}(y)$ is a column vector of dimension $m/2$ given by

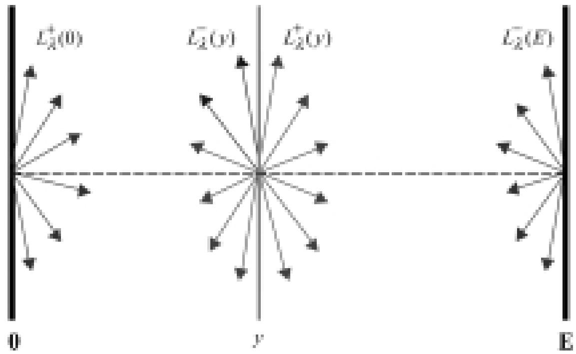

Figure 1.
Radiation intensity at boundaries and within the medium

$$\mathscr{E}_{\lambda}^{o,+}(y) = \left[\frac{\sigma_{a\lambda}(\mu_i)}{\mu_i} \cdot L_{\lambda}^{o}(T(y))\right]_{1 \leq i \leq m/2} \tag{19}$$

The symmetry properties, as in (Asllanaj *et al.*, 2000), lead to a constant square matrix A_λ of dimension m defined by

$$A_\lambda = \begin{bmatrix} A_\lambda^1 & A_\lambda^2 \\ -A_\lambda^2 & -A_\lambda^1 \end{bmatrix} \tag{20}$$

where A_λ^1 and A_λ^2 are two square matrices of dimension *m/2* of elements

$$\begin{aligned} (A_\lambda^1)_{ij} &= \frac{1}{\mu_i} \cdot \left\{ P_\lambda(\mu_j \to \mu_i) - \sigma_{e\lambda}(\mu_j) \cdot \delta_{ij} \right\} \quad \text{and} \\ (A_\lambda^2)_{ij} &= \frac{1}{\mu_i} \cdot P_\lambda(\mu_j \to -\mu_i) \end{aligned} \tag{21}$$

for *i* and *j* = 1.....*m/2* (*i* is the row index and *j* is the column index).

The monochromatic coefficients $\mathrm{P}_\lambda(\mu_\mathrm{j} \to \mu_\mathrm{i})$ take into account the scattering factors and the integration weights:

$$P_\lambda(\mu_j \to \mu_i) = \frac{1}{2} \cdot \sigma_s P_\lambda^*(\mu_j \to \mu_i) \cdot \sin\,(\theta_j) \cdot C_j \tag{22}$$

$0 < C_j < 1 \; \forall \, 1 \leq j \leq m$ and C_j are integration weights related to the numerical integration. θ_j are space polar directions such that $u_j = \cos(\theta_j)$with $\theta_j \in \;]0, \pi[\backslash \{\frac{\pi}{2}\}$.

In addition, from equations (4), (5) and (22)

$$\sum_{i=1}^{m} P_\lambda(\mu_j \to \mu_i) \cdot \frac{\sin\,(\theta_i) \cdot C_i}{\sin\,(\theta_j) \cdot C_j} = \sigma_{s\lambda}(\mu_j) + r \tag{23}$$

where r is a real, which represents the integration error. In our case, a composite

Newton-Côtes rule was used to discretize the integral term and $r = cte \cdot \triangle\theta^3$ where *cte* is a constant and $\triangle\theta = \frac{\pi}{m}$ is the constant integration step.

The discrete radiative boundary conditions are

$$\begin{aligned} L_\lambda(0, \mu_i) &= L^o_\lambda(T_o) \quad &for \quad 1 \le i \le m/2 \\ L_\lambda(E, -\mu_i) &= L^o_\lambda(T_E) \quad &for \quad 1 \le i \le m/2 \end{aligned} \tag{24}$$

The system (15) is associated with Dirichlet boundary conditions (Figure 1):

$$L^+_\lambda(0) = \begin{bmatrix} L^o_\lambda(T_o) \\ \vdots \\ L^o_\lambda(T_o) \end{bmatrix} \quad \text{and} \quad L^-_\lambda(E) = \begin{bmatrix} L^o_\lambda(T_E) \\ \vdots \\ L^o_\lambda(T_E) \end{bmatrix} \tag{25}$$

3.2 Space discretization

The first-order linear inhomogeneous ODE system with constant coefficients (15) can be solved in an analytical way. The general principle of resolution is well known: it consists in solving the associated homogeneous system and seeking a particular solution of the general system. The general solution of the system is obtained by summation of the two preceding solutions. Thus, let L_λ be the general solution of the system, $L_{h\lambda}$ the homogeneous solution of the system and $L_p\lambda$ a particular solution of the system. In the same way, we have for the boundary conditions (25)

$$\begin{cases} L^+_\lambda(0) = L^+_{h\lambda}(0) + L^+_{p\lambda}(0) \\ L^-_\lambda(E) = L^-_{h\lambda}(E) + L^-_{p\lambda}(E) \end{cases}$$

where $L^+_{h\lambda}(0)$ and $L^+_{p\lambda}(0)$ are homogeneous and particular boundary conditions at $y = 0$ respectively. $L^-_{h\lambda}(E)$ and $L^-_{p\lambda}(E)$ are homogeneous and particular boundary conditions at $y = E$ respectively.

3.2.1 Solution of the homogeneous system. It is about the radiative heat transfer problem without internal emission:

$$\frac{dL_{h\lambda(y)}}{dy} = A_\lambda \cdot L_{h\lambda}(y) \tag{26}$$

associated with the following radiative homogeneous boundary conditions:

$$\begin{cases} L^+_{h\lambda}(0) = L^+_\lambda(0) - L^+_{p\lambda}(0) \\ L^-_{h\lambda}(E) = L^-_\lambda(E) - L^-_{p\lambda}(E) \end{cases} \tag{27}$$

We point out that boundary conditions (27) are not homogeneous in general.

As is well-known, the solution of equation (26) together with the condition (27) may be written

$$L_{h\lambda}(y) = \exp(A_\lambda \cdot y) \cdot K_\lambda \tag{28}$$

where K_λ is a constant integration vector of dimension m determined according to the boundary conditions (27) and $\exp(A_\lambda \cdot y)$ is the exponential of the matrix $(A_\lambda \cdot y)$.

Let

$$\exp(A_\lambda \cdot y) = \begin{bmatrix} E^1_\lambda(y) & E^2_\lambda(y) \\ E^3_\lambda(y) & E^4_\lambda(y) \end{bmatrix} \quad \text{and} \quad K_\lambda = \begin{bmatrix} K^1_\lambda \\ K^2_\lambda \end{bmatrix} \tag{29}$$

where $E^i_\lambda(y)$ $1 \le i \le 4$ are square matrices of dimension $m/2$ and K^1_λ, K^2_λ two column vectors of dimension $m/2$. We then have

$$\begin{cases} L^+_{h\lambda}(y) = E^1_\lambda(y) \cdot K^1_\lambda + E^2_\lambda(y) \cdot K^2_\lambda \\ L^-_{h\lambda}(y) = E^3_\lambda(y) \cdot K^1_\lambda + E^4_\lambda(y) \cdot K^2_\lambda \end{cases} \quad \text{and}$$
$$\begin{cases} L^+_{h\lambda}(0) = K^1_\lambda \\ L^-_{h\lambda}(E) = E^3_\lambda(E) \cdot K^1_\lambda + E^4_\lambda(E) \cdot K^2_\lambda \end{cases} \tag{30}$$

We can therefore deduce the expression of the vector K_λ according to homogeneous radiative boundary conditions

$$K^1_\lambda = L^+_{h\lambda}(0) \quad \text{and} \quad K^2_\lambda = (E^4_\lambda(E))^{-1} \cdot (L^-_{h\lambda}(E) - E^3_\lambda(E) \cdot L^+_{h\lambda}(0)) \tag{31}$$

In addition, solution (28) has the semigroup property, i.e. if $0 < y_i < Y_{i+1} < E$ then

$$L_{h\lambda}(y_{i+1}) = \exp\ (A_\lambda \cdot (y_{i+1} - y_i)) \cdot L_h\ (y_i) \tag{32}$$

Thus, the solution at point y_{i+1} is directly given by the relation (32) if the solution at the point y_i is known.

However, the direct implementation of the solution calculation by relations (28) and (32), in the form of a computer code is impossible because it leads to a numerical instability. We will give the explanations by the elements which will follow. To start, we give some eigenvalue properties of the matrix A_λ.

Proposition 1: if ξ_λ is an eigenvalue of the matrix A_λ associated with the eigenvector $\begin{bmatrix} X^1_\lambda \\ X^2_\lambda \end{bmatrix}$ where X^1_λ, X^2_λ are two column vectors of dimension $m/2$, then $-\xi_\lambda$ is also an eigenvalue of the matrix A_λ associated with the eigenvector $\begin{bmatrix} X^2_\lambda \\ X^1_\lambda \end{bmatrix}$.

Proof: we have

$$\begin{bmatrix} A^1_\lambda & A^2_\lambda \\ -A^2_\lambda & -A^1_\lambda \end{bmatrix} \cdot \begin{bmatrix} X^1_\lambda \\ X^2_\lambda \end{bmatrix} = \xi_\lambda \cdot \begin{bmatrix} X^1_\lambda \\ X^2_\lambda \end{bmatrix}$$

or developing the product

$$\begin{cases} A^1_\lambda \cdot X^1_\lambda + A^2_\lambda \cdot X^2_\lambda = \xi_\lambda \cdot X^1_\lambda & (L_1) \\ -A^2_\lambda \cdot X^1_\lambda - A^1_\lambda \cdot X^2_\lambda = \xi_\lambda \cdot X^2_\lambda & (L_2) \end{cases}$$

which is equivalent to

$$\begin{cases} A^1_\lambda \cdot X^2_\lambda + A^2_\lambda \cdot X^1_\lambda = -\xi_\lambda \cdot X^2_\lambda & \text{multiplying } (L_2) \text{ by } -1 \\ -A^2_\lambda \cdot X^2_\lambda - A^1_\lambda \cdot X^1_\lambda = -\xi_\lambda \cdot X^1_\lambda & \text{multiplying } (L_1) \text{ by } -1 \end{cases}$$

i.e.

$$\begin{bmatrix} A^1_\lambda & A^2_\lambda \\ -A^2_\lambda & -A^1_\lambda \end{bmatrix} \cdot \begin{bmatrix} X^2_\lambda \\ X^1_\lambda \end{bmatrix} = -\xi_\lambda \cdot \begin{bmatrix} X^2_\lambda \\ X^1_\lambda \end{bmatrix} \quad \square$$

Proposition 2: if ξ_λ is an eigenvalue of the matrix A_λ associated with the eigenvector $\begin{bmatrix} X^1_\lambda \\ X^2_\lambda \end{bmatrix}$ where X^1_λ, X^2_λ are two column vectors of dimension $m/2$, then $(\xi_\lambda)^2$ is an eigenvalue of the matrix $(A^1_\lambda + A^2_\lambda) \cdot (A^1_\lambda - A^2_\lambda)$ associated with the eigenvector $(X^1_\lambda - X^2_\lambda)$ and is also an eigenvalue of the matrix $(A^1_\lambda - A^2_\lambda) \cdot (A^1_\lambda + A^2_\lambda)$ associated with the eigenvector $(X^1_\lambda + X^2_\lambda)$.

Proof: we have

$$\begin{bmatrix} A^1_\lambda & A^2_\lambda \\ -A^2_\lambda & -A^1_\lambda \end{bmatrix} \cdot \begin{bmatrix} X^1_\lambda \\ X^2_\lambda \end{bmatrix} = \xi_\lambda \cdot \begin{bmatrix} X^1_\lambda \\ X^2_\lambda \end{bmatrix}$$

Developing the product, we have

$$\begin{cases} A^1_\lambda \cdot X^1_\lambda + A^2_\lambda \cdot X^2_\lambda = \xi_\lambda \cdot X^1_\lambda & (L_1) \\ -A^2_\lambda \cdot X^1_\lambda - A^1_\lambda \cdot X^2_\lambda = \xi_\lambda \cdot X^2_\lambda & (L_2) \end{cases}$$

which is equivalent to

$$\begin{cases} (A^1_\lambda - A^2_\lambda) \cdot (X^1_\lambda - X^2_\lambda) = \xi_\lambda \cdot (X^1_\lambda + X^2_\lambda) & (L_1) := (L_1) + (L_2) \\ (A^1_\lambda + A^2_\lambda) \cdot (X^1_\lambda + X^2_\lambda) = \xi_\lambda \cdot (X^1_\lambda - X^2_\lambda) & (L_2) := (L_1) - (L_2) \end{cases}$$

Substituting (L_1) in (L_2) and (L_2) in (L_1), we obtain

$$\begin{cases} (A^1_\lambda + A^2_\lambda) \cdot (A^1_\lambda - A^2_\lambda) \cdot (X^1_\lambda - X^2_\lambda) = (\xi_\lambda)^2 \cdot (X^1_\lambda - X^2_\lambda) \\ (A^1_\lambda - A^2_\lambda) \cdot (A^1_\lambda + A^2_\lambda) \cdot (X^1_\lambda + X^2_\lambda) = (\xi_\lambda)^2 \cdot (X^1_\lambda + X^2_\lambda) \quad \square \end{cases}$$

Proposition 3: 0 is not an eigenvalue of the matrix A_λ.

Proof: let $\tilde{A}_\lambda$ be a square matrix of dimension m defined by $\tilde{A}_\lambda = G \cdot A_\lambda$ where G is a diagonal matrix of dimension m given by

$$G = \begin{pmatrix} \mu_1 \cdot \sin(\theta_1) \cdot C_1 & & & & & \\ & \ddots & & & 0 & \\ & & \mu_{m/2} \cdot \sin(\theta_{m/2}) \cdot C_{m/2} & & & \\ & & & -\mu_{m/2} \cdot \sin(\theta_{m/2}) \cdot C_{m/2} & & \\ & 0 & & & \ddots & \\ & & & & & -\mu_1 \cdot \sin(\theta_1) \cdot C_1 \end{pmatrix}$$

We first show that the matrix $\tilde{A}_\lambda$ is with a strictly dominant diagonal, i.e.

$\forall \ 1 \le j \le m \ , \ \left| (\tilde{A}_\lambda)_{jj} \right| \ > \sum_{\substack{i=1 \\ i \ne j}}^{m} \left| (\tilde{A}_\lambda)_{ij} \right|$. Indeed,

$|(\tilde{A}_\lambda)_{jj}| = |\sin(\theta_j) \cdot C_j| \cdot |\mathrm{P}_\lambda(\mu_j \to \mu_j) - \sigma_{e\lambda}(\mu_j)|$

$= (\sin(\theta_j) \cdot C_j) \cdot \{\sigma_{e\lambda}(\mu_j) - \mathrm{P}_\lambda(\mu_j \to \mu_j)\}$

because $0 < \sin(\theta_j) \cdot C_j < 1 \ \forall \ 1 \le j \le m, \sigma_{e\lambda}(\mu_j) = \sigma_{a\lambda}(\mu_j) + \sigma_{s\lambda}(\mu_j)$, from equation (3), $\sigma_{a\lambda}(\mu_j) > 0$ and $\mathrm{P}_\lambda(\mu_j \to \mu_j) < \sigma_{s\lambda}(\mu_j) \ \forall \ 1 \le j \le m$ from equations (4) and (22).

$$\sum_{\substack{i=1 \\ i \ne j}}^{m} \left| (\tilde{A}_\lambda)_{ij} \right| = \sum_{\substack{i=1 \\ i \ne j}}^{m} |\sin(\theta_i) \cdot C_i| \cdot | P_\lambda(\mu_j \to \mu_i) |$$

$$= \sum_{\substack{i=1 \\ i \ne j}}^{m} P_\lambda(\mu_j \to \mu_i) \cdot \sin(\theta_i) \cdot C_i$$

since P_λ is strictly positive from equation (22). In this case

$(\sin(\theta_j) \cdot C_j) \cdot \{ \ \sigma_{e\lambda}(\mu_j) - P_\lambda(\mu_j \to \mu_j) \ \} - \sum_{\substack{i=1 \\ i \ne j}}^{m} P_\lambda(\mu_j \to \mu_i) \cdot \sin(\theta_i) \cdot C_i$

$= \sin(\theta_j) \cdot C_j \cdot (\sigma_{a\lambda}(\mu_j) + \sigma_{s\lambda}(\mu_j)) - \sum_{i=1}^{m} P_\lambda(\mu_j \to \mu_i) \cdot \sin(\theta_i) \cdot C_i$

$= \sin(\theta_j) \cdot C_j \cdot (\sigma_{a\lambda}(\mu_j) - r) > 0$ from equation (23) because $r = cte \cdot \triangle\theta^3$ and $\exists \Delta\theta$ such that $\sigma_{a\lambda}(\mu_j) - r) > 0 \ \forall \ \lambda, j$.

The matrix $\tilde{A}_\lambda$ being with a strictly dominant diagonal, it is then invertible and det $(\tilde{A}_\lambda) \ne 0$.

In addition: det $(A_\lambda) = \det(G^{-1} \cdot \tilde{A}_\lambda) = \det(G^{-1}) \cdot \det(\tilde{A}_\lambda) \ne 0$ because det $(G^{-1}) = (-1)^{m/2} \cdot \prod_{i=1}^{m/2} \frac{1}{(\mu_i \cdot \sin(\theta_i) \cdot C_i)^2} \ne 0$ which proves that the matrix A_λ is invertible and therefore 0 is not an eigenvalue of this matrix. □

Remark 1: for the practical case, which interests us, we determine eigenvalues of the matrix A_λ using the Matlab software, which we use for the program. In our applications, we took $m = 12$ directions and $N\lambda = 211$ wavelengths. We represent on Figure 2, the $m/2$ positive eigenvalues of the matrix A_λ for each wavelength. Those are then real and simples for each wavelength.

The problem to be solved is mathematically well posed according to the Cauchy-Lipschitz theorem (Crouzeix and Mignot, 1992). However, it is numerically badly posed, since the solution is given by an exponential of the matrix from equation (28) and this matrix admits nonzero, real and positive eigenvalues, from the propositions 1, 3 and remark 1 (Crouzeix and Mignot, 1992). In addition, we point out that the direct calculation of the solution by the relation (28) will diverge very quickly because the eigenvalues of the matrix A_λ are very large, according to Figure 2.

In this section, we propose a new method for solving the homogeneous system (26) using a development based on the diagonalizing technique of the matrix A_λ.

If all the eigenvalues of the matrix A_λ are distinct, A_λ is diagonalizable. Then, according to remark 1, this is so in our particular case. By proposition 2,

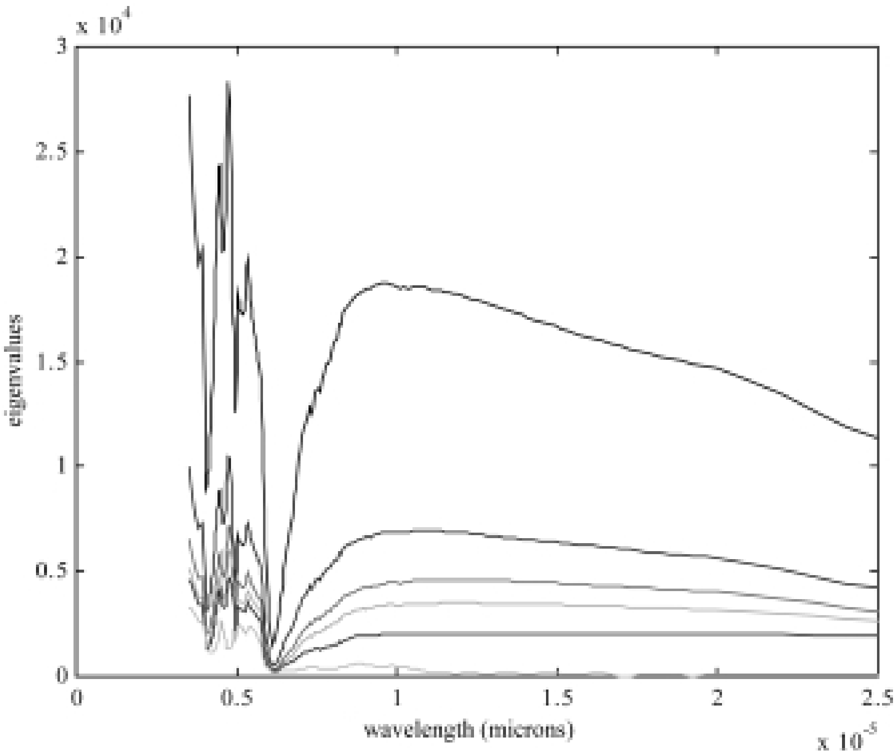

Figure 2.
Eigenvalues of the matrix A_λ

the matrix $(A^1_\lambda - A^2_\lambda)\cdot(A^1_\lambda + A^2_\lambda)$ is also diagonalizable. Its reduction is given by

$$(D_\lambda)^2 = S_\lambda^{-1}\cdot(A^1_\lambda - A^2_\lambda)\cdot (A^1_\lambda + A^2_\lambda)\cdot S_\lambda \tag{33}$$

where $(D_\lambda)^2$ is the diagonal matrix given by $(D_\lambda)^2 = diag\,[(\xi^j_\lambda)^2]_{1\le j\le m/2}$ and S_λ is the square passage matrix of dimension $m/2$. Reduction of the matrix A_λ is given by

$$A_\lambda = \frac{1}{4}\cdot\begin{bmatrix} U^1_\lambda & U^2_\lambda \\ U^2_\lambda & U^1_\lambda\end{bmatrix}\cdot\begin{bmatrix} D_\lambda & 0 \\ 0 & -D_\lambda\end{bmatrix}\cdot\begin{bmatrix} Z^1_\lambda & Z^2_\lambda \\ Z^2_\lambda & Z^1_\lambda\end{bmatrix} \tag{34}$$

where $U^1_\lambda = S_\lambda + W_\lambda$; $U^2_\lambda = S_\lambda - W_\lambda$; $Z^1_\lambda = (S_\lambda)^{-1} + (W_\lambda)^{-1}$; $Z^2_\lambda = (S_\lambda)^{-1} = -(W_\lambda)^{-1}$; $W_\lambda = (A^1_\lambda + A^2_\lambda)\cdot S_\lambda\cdot D_\lambda^{-1} = (A^1_\lambda - A^2_\lambda)^{-1}\cdot S_\lambda\cdot D_\lambda$

If we take again the solution given by equation (28)

$$L_{h\lambda}(y) = \exp(A_\lambda\cdot y)\cdot K_\lambda$$

and if we substitute the matrix A_λ by its expression given by equation (34), we have

$$L_h\,(y) = \frac{1}{4}\cdot\begin{bmatrix} U^1_\lambda & U^2_\lambda \\ U^2_\lambda & U^1_\lambda\end{bmatrix}\cdot\begin{bmatrix} \exp(D_\lambda\cdot y) & 0 \\ 0 & exp(-D_\lambda\cdot y)\end{bmatrix}\cdot\begin{bmatrix} Z^1_\lambda & Z^2_\lambda \\ Z^2_\lambda & Z^1_\lambda\end{bmatrix}\cdot\begin{bmatrix} K^1_\lambda \\ K^2_\lambda\end{bmatrix} \tag{35}$$

Setting

$$C_\lambda = \begin{bmatrix} C^1_\lambda \\ C^2_\lambda\end{bmatrix} = \frac{1}{4}\cdot\begin{bmatrix} Z^1_\lambda & Z^2_\lambda \\ Z^2_\lambda & Z^1_\lambda\end{bmatrix}\cdot\begin{bmatrix} K^1_\lambda \\ K^2_\lambda\end{bmatrix} \tag{36}$$

and

$$\exp_\lambda(y) = \begin{bmatrix} \tilde{\mathrm{E}}_\lambda(y) & 0 \\ 0 & \tilde{\mathrm{E}}_\lambda(-y)\end{bmatrix}\text{with } \tilde{\mathrm{E}}_\lambda(y) = \exp(D_\lambda\cdot y) \tag{37}$$

we then have

$$L_h\,(y) = \begin{bmatrix} U^1_\lambda & U^2_\lambda \\ U^2_\lambda & U^1_\lambda\end{bmatrix}\cdot\exp_\lambda(y)\cdot C_\lambda \tag{38}$$

Vectorial components of $L_{h\lambda}(y)$ corresponding to the front and back hemispheres are given by

$$\begin{aligned} L^+_{h\lambda}(y) &= [\,U^1_\lambda \quad U^2_\lambda]\cdot\exp_\lambda(y)\cdot C_\lambda \quad \text{and} \\ L^-_{h\lambda}(y) &= [\,U^2_\lambda \quad U^1_\lambda]\cdot\exp_\lambda(y)\cdot C_\lambda \end{aligned} \tag{39}$$

C_λ is a constant vector of dimension m determined by the boundary conditions

$$L^+_{h\lambda}(0) = [\, U^1_\lambda \quad U^2_\lambda] \cdot\; C_\lambda \;\text{ and }\; L^-_{h\lambda}(E) = [\, U^2_\lambda \quad U^1_\lambda] \cdot \exp_\lambda(E) \cdot C_\lambda \qquad (40)$$

or

$$\begin{bmatrix} [U^1_\lambda \quad U^2_\lambda] \\ [U^2_\lambda \quad U^1_\lambda] \cdot \exp_\lambda(E) \end{bmatrix} \cdot\; C_\lambda = \begin{bmatrix} L^+_{h\lambda}(0) \\ L^-_{h\lambda}(E) \end{bmatrix} \qquad (41)$$

Substituting C_λ given by equation (41) in equation (38), we obtain the radiation intensity expression according to the boundary conditions

$$L_{h\lambda}(y) = \begin{bmatrix} U^1_\lambda & U^2_\lambda \\ U^2_\lambda & U^1_\lambda \end{bmatrix} \cdot\; \exp_\lambda(y) \cdot \begin{bmatrix} [U^1_\lambda \quad U^2_\lambda] \\ [U^2_\lambda \quad U^1_\lambda] \cdot \exp_\lambda(E) \end{bmatrix}^{-1} \cdot \begin{bmatrix} L^+_{h\lambda}(0) \\ L^-_{h\lambda}(E) \end{bmatrix} \qquad (42)$$

Remark 2: the matrix $\exp_\lambda$ has the following properties:

$$(\exp_\lambda(y))^{-1} = \exp_\lambda(-y) \;\text{ and }\; \exp_\lambda(x) \cdot \exp(y) = \exp_\lambda(x+y)$$

Then using $\exp_\lambda$ properties and recalling that

$$B^{-1} \cdot A^{-1} = (A \cdot B)^{-1} \qquad (43)$$

we have:

$$L_{h\lambda}(y) = \begin{bmatrix} U^1_\lambda & U^2_\lambda \\ U^2_\lambda & U^1_\lambda \end{bmatrix} \cdot \begin{bmatrix} [\, U^1_\lambda \quad U^2_\lambda] \cdot\; \exp_\lambda(-y) \\ [\, U^2_\lambda \quad U^1_\lambda] \cdot \exp_\lambda(E-y) \end{bmatrix}^{-1} \cdot \begin{bmatrix} L^+_{h\lambda}(0) \\ L^-_{h\lambda}(E) \end{bmatrix} \qquad (44)$$

Developing the matrix expression which must be inverted, we have:

$$L_{h\lambda}(y) = \begin{bmatrix} U^1_\lambda & U^2_\lambda \\ U^2_\lambda & U^1_\lambda \end{bmatrix} \cdot \begin{bmatrix} U^1_\lambda \cdot \tilde{\mathrm{E}}_\lambda(-y) & U^2_\lambda \cdot \tilde{\mathrm{E}}_\lambda(y) \\ U^2_\lambda \cdot \tilde{\mathrm{E}}_\lambda(E-y) & U^1_\lambda \cdot \tilde{\mathrm{E}}_\lambda(y-E) \end{bmatrix}^{-1} \cdot \begin{bmatrix} L^+_{h\lambda}(0) \\ L^-_{h\lambda}(E) \end{bmatrix} \qquad (45)$$

$\tilde{\mathrm{E}}_\lambda(a) diag\ (e^{\xi^j_\lambda \cdot a})_{1 \le j \le m/2}$ where $\xi^j_\lambda 1 \le j \le m/2$, are the strictly positive eigenvalues of the matrix A_λ. Hence, $\tilde{\mathrm{E}}_\lambda(a)$diverges if a is positive. In order to determine the matrix of equation (45) which must be inverted, we will consider the matrix $\tilde{\mathrm{E}}_\lambda(a)$ where a will be a negative value.

Remark 3: the matrix $\tilde{\mathrm{E}}_\lambda$ has the same properties as the matrix $\exp_\lambda$.

In this case,

$$\begin{bmatrix} U_\lambda^1 \cdot \tilde{E}_\lambda(-y) & U_\lambda^2 \cdot \tilde{E}_\lambda(y) \\ U_\lambda^2 \cdot \tilde{E}_\lambda(E-y) & U_\lambda^1 \cdot \tilde{E}_\lambda(y-E) \end{bmatrix}^{-1}$$
$$= \left\{ \begin{bmatrix} U_\lambda^1 \cdot \tilde{E}_\lambda(-E) & U_\lambda^2 \\ U_\lambda^2 & U_\lambda^1 \cdot \tilde{E}_\lambda(-E) \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_\lambda(E-y) & 0 \\ 0 & \tilde{E}_\lambda(y) \end{bmatrix} \right\}^{-1}$$
$$= \begin{bmatrix} \tilde{E}_\lambda(y-E) & 0 \\ 0 & \tilde{E}_\lambda(-y) \end{bmatrix} \cdot$$
$$\begin{bmatrix} U_\lambda^1 \cdot \tilde{E}_\lambda(-E) & U_\lambda^2 \\ U_\lambda^2 & U_\lambda^1 \cdot \tilde{E}_\lambda(-E) \end{bmatrix}^{-1} \text{using equation (43).} \tag{46}$$

Let

$$B_\lambda^1(y) = \tilde{\mathrm{E}}_\lambda(y-E) \;\text{ and }\; B_\lambda^2(y) = \tilde{\mathrm{E}}_\lambda(-y) \tag{47}$$

Putting equation (46) in equation (45) and using equation (47), we obtain finally the expression which makes it possible to calculate the radiation intensity at any point of the medium without any numerical problem:

$$L_{h\lambda}(y) = \begin{bmatrix} U_\lambda^1 & U_\lambda^2 \\ U_\lambda^2 & U_\lambda^1 \end{bmatrix} \cdot \begin{bmatrix} B_\lambda^1(y) & 0 \\ 0 & B_\lambda^2(y) \end{bmatrix} \cdot \begin{bmatrix} U_\lambda^1 \cdot \tilde{E}_\lambda(-E) & U_\lambda^2 \\ U_\lambda^2 & U_\lambda^1 \cdot \tilde{\mathrm{E}}_\lambda(-E) \end{bmatrix}^{-1} \cdot \begin{bmatrix} L_{h\lambda}^+(0) \\ L_{h\lambda}^-(E) \end{bmatrix} \tag{48}$$

Remark 4:

(1) The matrix $B_\lambda^i \; i = 1, 2$ can be calculated by iteration. Indeed, let $0 < y_i < y_{i+1} < E$, then $B_\lambda^1(y_i) = \tilde{E}_\lambda(-(y_{i+1} - y_i)) \cdot b_\lambda^1(y_{i+1})$ and $b_\lambda^2(y_{1+1}) = \tilde{E}_\lambda(-(y_{i+1} - y_i)) \cdot B_\lambda^2(y_i)$.

(2) In our application, generally $\tilde{\mathrm{E}}_\lambda(-\mathrm{E}) = 0$, the zero matrix (see Figure 2). In this case,

$$\begin{bmatrix} U_\lambda^1 \cdot \tilde{\mathrm{E}}_\lambda(-E) & U^2 \\ U_\lambda^2 & U_\lambda^1 \cdot \tilde{\mathrm{E}}_\lambda(-E) \end{bmatrix}^{-1} = \begin{bmatrix} 0 & U_\lambda^2 \\ U_\lambda^2 & 0 \end{bmatrix} = \begin{bmatrix} o & (U_\lambda^2)^{-1} \\ (U_\lambda^2)^{-1} & 0 \end{bmatrix}.$$

(3) The reduction of the matrix A_λ is brought back in fact to the reduction of the matrix $(A_\lambda^1 - A_\lambda^2) \cdot (A_\lambda^1 + A_\lambda^2)$, which reduces the order of the problem by two. The Matlab software that we use, gives eigenvalues

and eigenvectors of a matrix rather quickly and with very good precision. The calculation of the matrix $\tilde{E}_\lambda$ does not pose a problem since it is given analytically by the relation

$$\tilde{E}_\lambda(-x) = diag(e^{-\xi_\lambda^j \cdot x})_{1 \leq j \leq m/2} \; \forall x \in [0, E].$$

3.2.2 Particular solution of the system. We now seek a particular solution to the system with second member:

$$\frac{dL_\lambda(y)}{dy} = A_\lambda \cdot L_\lambda(y) + \mathscr{E}^o_\lambda(y) \tag{49}$$

Since no obvious solution appears, we use the variation method of the constants, i.e. we seek a particular solution in the form

$$L_{p\lambda}(y) = \exp(A_\lambda \cdot y) \cdot K_\lambda(y) \tag{50}$$

where K_λ is supposed to be differentiable. K_λ is given by

$$K_\lambda(y) = \int_0^y \exp(-A_\lambda \cdot u) \cdot \mathscr{E}^o_\lambda(u)\, du + C_\lambda \; \forall \; 0 < y < E \tag{51}$$

where C_λ is an unspecified constant vector of dimension m. Generally we fix $C_\lambda = 0$, but here, C_λ plays a significant role and we will choose it in order to have no numerical instability problem. Then, the particular solution is given by

$$L_{p\lambda}(y) = \int_0^y \exp(A_\lambda \cdot (y-u)) \cdot \mathscr{E}^o_\lambda(u)\, du + \exp(A_\lambda \cdot y) \cdot C_\lambda \tag{52}$$

If we take again the reduced form of the matrix A_λ given by equation (34) and if we apply it to equation (52), we have

$$L_{p\lambda}(y) = \frac{1}{4} \cdot \begin{bmatrix} U_\lambda^1 & U_\lambda^2 \\ U_\lambda^2 & U_\lambda^1 \end{bmatrix} \cdot \int_0^y \begin{bmatrix} \exp(D_\lambda \cdot (y-u)) & 0 \\ 0 & \exp(D_\lambda \cdot (u-y)) \end{bmatrix} \cdot \begin{bmatrix} Z_\lambda^1 & Z_\lambda^2 \\ Z_\lambda^2 & Z_\lambda^1 \end{bmatrix} \cdot \mathscr{E}^o_\lambda(u)\; du + \frac{1}{4} \cdot \begin{bmatrix} U_\lambda^1 & U_\lambda^2 \\ U_\lambda^2 & U_\lambda^1 \end{bmatrix} \cdot \begin{bmatrix} \exp(D_\lambda \cdot y) & 0 \\ 0 & \exp(-D_\lambda \cdot y) \end{bmatrix} \cdot \begin{bmatrix} Z_\lambda^1 & Z_\lambda^2 \\ Z_\lambda^2 & Z_\lambda^1 \end{bmatrix} \cdot C_\lambda \tag{53}$$

Let

$$\tilde{\mathscr{E}}^{o}_{\lambda}(u) = \begin{bmatrix} \tilde{\mathscr{E}}^{o,+}_{\lambda}(u) \\ \tilde{\mathscr{E}}^{o,-}_{\lambda}(u) \end{bmatrix} = \frac{1}{4} \cdot \begin{bmatrix} Z^1_\lambda & Z^2_\lambda \\ Z^2_\lambda & Z^1_\lambda \end{bmatrix} \cdot \mathscr{E}^{o}_{\lambda}(u) \tag{54}$$

and

$$\tilde{C}_\lambda = \begin{bmatrix} \tilde{C}^+_\lambda \\ \tilde{C}^-_\lambda \end{bmatrix} = \frac{1}{4} \cdot \begin{bmatrix} Z^1_\lambda & Z^2_\lambda \\ Z^2_\lambda & Z^1_\lambda \end{bmatrix} \cdot C_\lambda \tag{55}$$

Then $\tilde{\mathscr{E}}^{o,-}_{\lambda}(u) = -\tilde{\mathscr{E}}^{o,+}_{\lambda}(u)$ since $\mathscr{E}^{o}_{\lambda}(u) = \begin{bmatrix} \mathscr{E}^{o,+}_{\lambda}(u) \\ -\mathscr{E}^{o,+}_{\lambda}(u) \end{bmatrix}$ from equation (18).

Using equations (54) and (55) in equation (53), we obtain

$$L_{p\lambda}(y) = \begin{bmatrix} U^1_\lambda & U^2_\lambda \\ U^2_\lambda & U^1_\lambda \end{bmatrix} \cdot \begin{bmatrix} \int_0^y \tilde{E}_\lambda(y-u) \cdot \tilde{\mathscr{E}}^{o,+}_{\lambda}(u)\, du \ + \tilde{E}_\lambda(y) \cdot \tilde{C}^+_\lambda \\ -\int_0^y \tilde{E}_\lambda(u-y) \cdot \tilde{\mathscr{E}}^{o,+}_{\lambda}(u)\, du \ + \tilde{E}_\lambda(-y) \cdot \tilde{C}^-_\lambda \end{bmatrix} \tag{56}$$

with $\tilde{\mathrm{E}}_\lambda(y) = \exp(D_\lambda \cdot y)$, the matrix defined by equation (37).

$\tilde{C}^+_\lambda$ are $\tilde{C}^-_\lambda$ constant vectors of dimension $m/2$ that we can even fix. Then, let $\tilde{C}^+_\lambda = -\int_0^E \tilde{E}_\lambda(-u) \cdot \mathscr{E}^{o,+}_{\lambda}(u)\, du$ and $\tilde{C}^-_\lambda = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ the null column vector of dimension $m/2$.

Hence

$$\tilde{E}_\lambda(y) \cdot \tilde{C}^+_\lambda = -\int_0^y \tilde{E}_\lambda(y-u) \cdot \tilde{\mathscr{E}}^{o,+}_{\lambda}(u)\, du - \int_y^E \tilde{\mathrm{E}}_\lambda(y-u) \cdot \tilde{\mathscr{E}}^{o,+}_{\lambda}(u)\, du$$

using the properties $\tilde{\mathrm{E}}_\lambda$ of and cutting the integral into two. In this case

$$L_{p\lambda}(y) = \begin{bmatrix} L^+_{p\lambda}(y) \\ L^-_{p\lambda}(y) \end{bmatrix} = -\begin{bmatrix} U^1_\lambda & U^2_\lambda \\ U^2_\lambda & U^1_\lambda \end{bmatrix} \cdot \begin{bmatrix} \int_y^E \tilde{E}_\lambda(y-u) \cdot \tilde{\mathscr{E}}^{o,+}_{\lambda}(u)\, du \\ \int_0^y \tilde{E}_\lambda(u-y) \cdot \tilde{\mathscr{E}}^{o,+}_{\lambda}(u)\, du \end{bmatrix} \tag{57}$$

or developing the product, we obtain the components for each hemisphere

$$L_{p\lambda}^{+}(y) = -U_{\lambda}^{1} \cdot \int_{y}^{E} \tilde{E}_{\lambda}(y-u) \cdot \tilde{\mathscr{E}}_{\lambda}^{o,+}(u) \ \ du - \ U_{\lambda}^{2} \cdot \int_{0}^{y} \tilde{E}_{\lambda}(u-y) \cdot \tilde{\mathscr{E}}_{\lambda}^{o,+}(u) \ du$$

$$L_{p\lambda}^{-}(y) = -U_{\lambda}^{2} \cdot \int_{y}^{E} \tilde{E}_{\lambda}(y-u) \cdot \tilde{\mathscr{E}}_{\lambda}^{o,+}(u) \ \ du - \ U_{\lambda}^{1} \cdot \int_{0}^{y} \tilde{E}_{\lambda}(u-y) \cdot \tilde{\mathscr{E}}_{\lambda}^{o,+}(u) \ du \tag{58}$$

“particular” boundary conditions are then given by

$$L_{p\lambda}^{+}(0) = - \ U_{\lambda}^{1} \cdot \int_{0}^{E} \tilde{E}_{\lambda}(-u) \cdot \tilde{\mathscr{E}}_{\lambda}^{o,+}(u) \ \ du \ ; \ L_{p\lambda}^{-}(E) = - \ U_{\lambda}^{1} \cdot \int_{0}^{E} \tilde{E}_{\lambda}(u-E) \cdot \tilde{\mathscr{E}}_{\lambda}^{o,+}(u) \ du \tag{59}$$

We realise then:

- when $y < u < E$, the calculation of $\tilde{E}_{\lambda}(y-u)$ is stable, since $y-u<0$,
- when $0 < u < y$, the calculation of $\tilde{E}_{\lambda}(u-y)$ is stable, since $u-y<0$.

Thus, the computation of the particular solution and the “particular” boundary conditions given respectively by relations (57) and (59) lead to a stable numerical problem.

Let

$$(a) = \int_{y}^{E} \tilde{\mathrm{E}}_{\lambda}(y-u) \cdot \tilde{\mathscr{E}}_{\lambda}^{o,+}(u) \ du \text{ with } 0 \le y < E \tag{60}$$

and

$$(b) = \int_{0}^{y} \tilde{\mathrm{E}}_{\lambda}(u-y) \cdot \tilde{\mathscr{E}}_{\lambda}^{o,+}(u) \ du \text{ with } 0 \le y < E \tag{61}$$

From equations (18), (19) and (54), $\tilde{\mathscr{E}}_{\lambda}^{o,+}$ is then a column vector of dimension $m/2$ given by

$$\tilde{\mathscr{E}}_{\lambda}^{o,+}(u) \ = \frac{1}{4}(Z_{\lambda}^{1} - Z_{\lambda}^{2}) \cdot \left[\frac{\sigma_{a\lambda}(\mu_i)}{u_i} \cdot L_{\lambda}^{o}(T(u)) \right]_{1 \le i \le m/2} \tag{62}$$

Thus, putting equation (62) in equations (60) and (61), we have

$$(a) = \frac{1}{4}(Z_\lambda^1 - Z_\lambda^2) \cdot Int1(y) \text{ with } 0 \leq y < E \tag{63}$$

with

$$(b) = \frac{1}{4}(Z_\lambda^1 - Z_\lambda^2) \cdot Int2(y) \text{ with } 0 \leq y < E \tag{64}$$

where $Int1(y)$ and $Int2(y)$ are two column vectors of dimension $m/2$ given by

$$Int1(y) = \left[\frac{\sigma_{a\lambda}(\mu_i)}{\mu_i} \cdot \int_y^E e^{D_i \cdot (y-u)} \cdot L_\lambda^o(T(u))\, du\right]_{1 \leq i \leq m/2} \tag{65}$$

$$Int2(y) = \left[\frac{\sigma_{a\lambda}(\mu_i)}{\mu_i} \cdot \int_0^y e^{D_i \cdot (u-y)} \cdot L_\lambda^o(T(u))\, du\right]_{1 \leq i \leq m/2} \tag{66}$$

and $(D_i)_{1 \leq i \leq m/2}$ are the elements of the diagonal matrix D.

Finally, putting equations (63) and (64) in equation (57), the particular solution is given by

$$L_{p\lambda}(y) = -U_\lambda \cdot \begin{bmatrix} \frac{1}{4}(Z_\lambda^1 - Z_\lambda^2) \cdot Int1(y) \\ \frac{1}{4}(Z_\lambda^1 - Z_\lambda^2) \cdot Int2(y) \end{bmatrix} \tag{67}$$

and "particular" boundary conditions by

$$\begin{aligned} L_{p\lambda}^+(0) &= -U_\lambda^1 \cdot \frac{1}{4}(Z_\lambda^1 - Z_\lambda^2) \cdot Int1(0) \text{ and} \\ L_{p\lambda}^-(E) &= -U_\lambda^1 \cdot \frac{1}{4}(Z_\lambda^1 - Z_\lambda^2) \cdot Int2(E) \end{aligned} \tag{68}$$

Now, we will calculate the integrals (for an index $1 \leq i \leq m/2$ fixed) given by

$$\int_y^E e^{D_i \cdot (y-u)} \cdot L_\lambda^0(T(u))\, du \text{ with } 0 \leq y \leq E \tag{69}$$

$$\int_0^y e^{D_i \cdot (u-y)} \cdot L_\lambda^0(T(u))\, du \text{ with } 0 \leq y \leq E \tag{70}$$

To do so, the spatial domain [0, E] is discretized (with a constant step or not) setting

$$0 = y_0 < y_1 < y_2 < \ldots\ldots < y_{nt} < y_{nt+1} = E.$$

Let

$$G_j^i = \int_{y_j}^{E} e^{D_i\cdot(y_j-u)} \cdot L_\lambda^o(T(u)) \;\; du \quad \text{with} \quad 0 \le j \le nt \tag{71}$$

$$K_j^i = \int_{0}^{y_j} e^{D_i\cdot(u-y_j)} \cdot L_\lambda^o(T(u)) \;\; du \quad \text{with } 1 \le j \le nt+1 \tag{72}$$

Then, the integrals are determined by recurrence:

$$G_j^i = \int_{y_j}^{y_{j+1}} e^{D_i\cdot(y_j-u)} \cdot L_\lambda^o(T(u))\; du \; + \; e^{D_i\cdot(y_j-y_{j+1})} \cdot G_{j+1}^i \quad 0 \le j \le nt-1 \tag{73}$$

$$K_{j+1}^i = \; e^{D_i\cdot(y_j-y_{j+1})} \cdot K_j^i \;\; + \int_{y_j}^{y_{j+1}} e^{D_i\cdot(u-y_{j+1})} \cdot L_\lambda^o(T(u)) \;\; du \quad 1 \le j \le nt \tag{74}$$

$L_\lambda^o(T(u))$ is known only at points $u = y_j$ and $u = y_{j+1}$ on the interval $[y_j, y_{j+1}]$, it is then approximated by a line on this interval, let $L_\lambda^o(T(u)) \cong a_j \cdot u + b_j$. Then

$$\int_{y_j}^{y_{j+1}} e^{D_i\cdot(y_j-u)} \cdot L_\lambda^o(T(u)) \;\; du \cong \int_{y_j}^{y_{j+1}} e^{D_i\cdot(y_j-u)} \cdot (a_j \cdot u + b_j) \;\; du$$
$$= \frac{1}{D_i} \cdot \left[a_j \cdot (\frac{1}{D_i} + y_j) + b_j - (a_j \cdot (\frac{1}{D_i} + y_{j+1}) + b_j) \cdot e^{D_i\cdot(y_j-y_{j+1})} \right]$$

and

$$\int_{y_j}^{y_{j+1}} e^{D_i\cdot(u-y_{j+1})} \cdot L_\lambda^o(T(u)) \;\; du \cong \int_{y_j}^{y_{j+1}} e^{D_i\cdot(u-y_{j+1})} \cdot (a_j \cdot u + b_j) \;\; du$$
$$= \frac{1}{D_i} \cdot \left[a_j \cdot (y_{j+1} - \frac{1}{D_i}) + b_j - (a_j \cdot (y_j - \frac{1}{D_i}) + b_j) \cdot e^{D_i\cdot(y_j-y_{j+1})} \right]$$

3.3 Spectral discretization

We are also concerned with the spectral discretization of the equation. λ is a parameter and the system (15) must be solved for each wavelength λ. For the resolution, we took account of the contribution of all significant wavelengths that we denote $\lambda_j 1 \leq j \leq N_\lambda$. Finally, we need to solve

$$\frac{dL_{\lambda_j}(y)}{dy} = A_{\lambda_j} \cdot L_{\lambda_j}(y) + \mathscr{E}^o_{\lambda_j}(y) \quad \forall\, 1 \leq j \leq N_\lambda \tag{75}$$

4. Coupling with conduction and results

The numerical scheme of the nonlinear heat conduction given by equations (7) and (8) is described in Asllanaj *et al.* (2000). Thus, the details of this procedure will not be repeated here. The main features of this technique are:

(1) The Kirchhoff transformation applied to the equation and the boundary conditions.

(2) Resolution of a set of one real variable nonlinear equations.

(3) Resolution of the one-dimensional linear Laplace equation with Dirichlet boundary conditions, by a finite difference method of order two.

The overall resolution scheme of the coupled equations, the calculation of the integral given by equation (10) and the calculation of the derivatives which intervene in equations (9) and (11) are also described in Asllanaj *et al.* (2000).

The method has been tested on a real fibrous material composed of silica fibers. Fibers have a diameter of seven microns and are randomly oriented in parallel planes to the boundaries. This material is the same as that in Asllanaj *et al.* (2000). The application is carried out under the same conditions, that we recall:

$E = 10$ cm	medium thickness
$\rho = 20$ kg/m^3	medium density
$T_o = 400$ K and $T_E = 300$ K	boundary temperature
$h = \frac{E}{nt+1}$ with $nt = 100$	constant space mesh
$m = 12$	angular discretization number
$N\lambda = 211$	spectral discretization number
$\varepsilon = 10^{-6}$	tolerance.

The present results have been compared with those obtained by the second numerical method using the finite difference method for the resolution in space of the RTE and with those obtained by the third method using transmission and reflection matrices for the resolution in space of the RTE (Asllanaj *et al.*, 2000). Table I presents a comparison between the three numerical methods. For each entry, three values are presented: on the right, the values for the new method, in the center for the second and on the left for the third method. The

Thickness (mm)	Total heat flux ($W \cdot m^{-2}$)	Radiative flux ($W \cdot m^{-2}$)	Conductive flux ($W \cdot m^{-2}$)	Temperature (K)
0				400.00 / 400.00 / 400.00
10	47.83 / 47.83 / 47.82	20.45 / 20.41 / 20.44	27.39 / 27.42 / 27.39	391.50 / 391.47 / 391.45
20	47.83 / 47.83 / 47.81	20.06 / 20.08 / 20.05	27.77 / 27.74 / 27.76	383.24 / 383.20 / 383.19
30	47.83 / 47.83 / 47.81	19.41 / 19.39 / 19.40	28.42 / 28.43 / 28.41	374.66 / 374.63 / 374.61
40	47.83 / 47.83 / 47.81	18.63 / 18.62 / 18.62	29.20 / 29.20 / 29.19	365.70 / 365.66 / 365.66
50	47.83 / 47.83 / 47.81	17.74 / 17.74 / 17.74	30.08 / 30.07 / 30.07	356.29 / 356.26 / 356.25
60	47.83 / 47.83 / 47.81	16.78 / 16.76 / 16.77	31.05 / 31.06 / 31.04	346.37 / 346.35 / 346.35
70	47.83 / 47.83 / 47.81	15.73 / 15.74 / 15.73	32.09 / 32.08 / 32.08	335.88 / 335.87 / 335.87
80	47.83 / 47.83 / 47.81	14.61 / 14.61 / 14.60	33.22 / 33.21 / 33.21	324.76 / 324.76 / 324.76
90	47.83 / 48.83 / 47.81	13.31 / 13.31 / 13.31	34.51 / 34.52 / 34.50	312.91 / 312.91 / 312.92
100				300.00 / 300.00 / 300.00

Table I. Flux variations and temperature distribution within the medium

results indicated show that the three methods are always in perfect agreement. This comparison shows that the method generates reliable results. Moreover, we have recorded computing times and the present method is approximately six times faster than the third method. We point out that the second method was approximately five times faster than the third method. The present method is therefore fastest. In addition, various tests proved that the coupled scheme is stable.

We represent in Figures 3 and 4 respectively, the temperature field and radiative, conductive heat flux, total heat flux, according to the position in the medium.

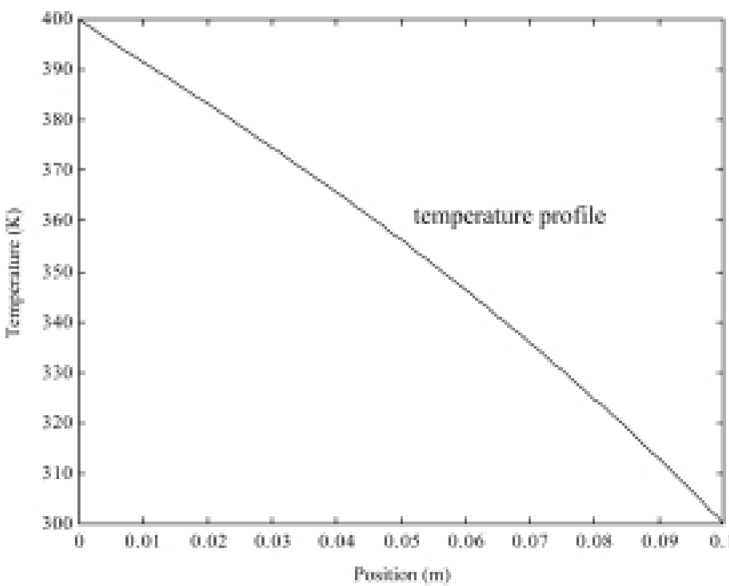

Figure 3. Temperature distribution within the medium

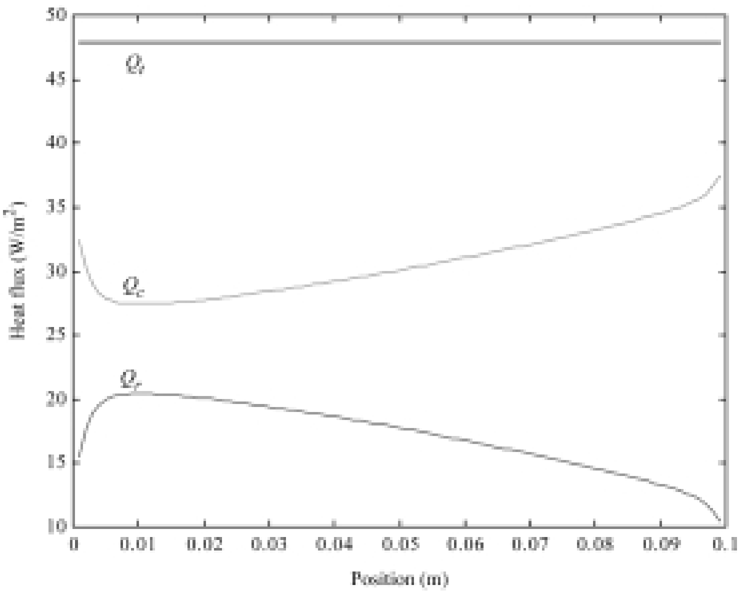

Figure 4.
Heat flux within the medium

5. Discussion

In the transient state, the radiative transfer equation takes the form:

$$\frac{1}{c} \cdot \frac{\partial L_\lambda(y,\mu,t)}{\partial t} + \mu \cdot \frac{\partial L_\lambda(y,\mu,t)}{\partial y} = \sigma_{a\lambda}(\mu) \cdot L^o_\lambda(T(y,t)) - \sigma_{e\lambda}(\mu) \cdot L_\lambda(y,\mu,t) + \frac{1}{2} \cdot \int_{\mu'=-1}^{1} \sigma_s P^*_\lambda(\mu' \to \mu) \cdot L_\lambda(y,\mu',t) \ d\mu'$$

where c denotes the radiation propagation speed in the medium (c = $2.997930 \cdot 10^8$ m/s) and t is the time. For most engineering applications, the term $\frac{1}{c} \cdot \frac{\partial L_\lambda}{\partial t}$ in the equation can be neglected in comparison to the other terms because of the large magnitude of propagation speed c (see (Ozisik, 1973)). The equation simplifies to

$$\mu \cdot \frac{\partial L_\lambda(y,\mu,t)}{\partial y} = \sigma_{a\lambda}(\mu) \cdot L^o_\lambda(T(y,t)) - \sigma_{e\lambda}(\mu) \cdot L_\lambda(y,\mu,t) + \frac{1}{2} \cdot \int_{\mu'=-1}^{1} \sigma_s P^*_\lambda(\mu' \to \mu) \cdot L_\lambda(y,\mu',t) \ d\mu'$$

The RTE is then quasi-stationary. In other words, time occurs in the equation only through the emission term by the intermediary of the temperature field. The time variable is considered merely as a parameter. The dependence of radiation intensity in time is then implicit through the temperature. The temperature field evolution in the transient-state will be described by the heat equation. Hence, the resolution in the transient-state of the RTE will be the same as in the steady-state. Finally, the method developed for the resolution of the RTE in this paper could be used directly in the transient-state. The principal interest of this method is that it is very fast and it can be used with a non-uniform mesh in space without any problem, since it gives an analytical solution in space. Let us note however that the method is valid only when the medium has symmetry and leads to a structure of the characteristic matrix A_λ as in equation (20).

The further development of this study now will relate to the transient-state with temperature boundary conditions, which vary very quickly in time and from very intense radiative flux boundary conditions. This work will use a non-uniform adaptive mesh.

6. Conclusion

A new resolution method of the radiative transfer equation for a non-grey participating media with anisotropic scattering has been presented. The resolution is based on the diagonalizing technique of the medium characteristic matrix which made it possible to circumvent the numerical instability problem. The method has proven to be in agreement with numerical solutions computed by other methods. The calculations are efficient in terms of computation times and the resolution is analytical in space. With this new model, we may now undertake the transient-state study.

This work is the result of a collaboration between the LEMTA Laboratory and the IECN Institute, both of the University of Nancy I, and the Isover Saint Gobain Company. This collaboration will continue within the framework of the transient-state study.

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